Composite Plates Under Concentrated Load on One Edge and Uniform Load on the Opposite Edge

Christos Kassapoglou¹ and George Bauer²
¹Department of Aerospace Engineering, Delft University of Technology, Delft, Netherlands
²CAE Associates, Inc., Middlebury, CT, USA

An approach to determine the stresses in a composite plate under a concentrated load on one edge and uniform load on the opposite edge is presented. It uses an exact solution of the governing partial differential equation obtained with the use of Fourier series. The solution is compared to finite element results and is found to be in excellent agreement. The solution to this problem can be used in the analysis of the effect of a stiffener termination in a panel or the transition from a stiffened panel to a sandwich or monolithic construction. Examples of dimensions and layup required to transfer the concentrated load into a composite plate are given.

Keywords Polymer-matrix composites, plate theory, modeling, stress transfer

1. INTRODUCTION

The design requirements of composite plates or shells in airframe structures often include transition of a stiffened design to a monolithic or sandwich design. The most common example is that of a stiffener termination [1–3]. The local stiffness mismatch and eccentricities due to the termination, lead to localized bending and in-plane loads that may result in premature failure of stiffened panels. In other cases [4], grids in grid-stiffened panels flare out into the skin of the panel to facilitate attaching to adjacent panels. Transitioning the load from the axial member into a non-stiffened plate causes a local stress concentration which must be properly accounted for during design to avoid failure. In all these cases, both membrane and bending loads act on the plate beyond the stiffener termination and a complete analysis would have to account for both. As a first step, an analysis to determine the membrane loads is presented here.

The situation is shown in Figure 1. A concentrated load is exerted on one edge of a composite plate. The width over which the load is distributed is small compared to the width of the plate and can represent the flange width of a stiffener that terminates at the edge of the plate. The axial load in the stiffener is transferred into the plate. The concentrated load is reacted at the other end of the plate by a uniform load. Note that the simplified problem at the bottom of Figure 1 will not model the stiffened panel transferring load into a flat panel accurately if the stiffeners are so close that the stress field in the flat plate caused by one stiffener interacts with that of adjacent stiffeners. This limitation will be further investigated later.

2. CLOSED FORM SOLUTION

A discussion of the problem of Figure 1 for isotropic plates can be found in Timoshenko and Goodier [5]. There, Fourier series are used to solve a case where a concentrated load acts on both ends of an isotropic plate and the case of a uniform distributed load on one end is obtained at the center of the plate in the limiting case where the plate is very long. Here, an analogous approach is used but with the added complication of material anisotropy.

With reference to Figure 1, and basic mechanics of anisotropic plates, see for example [6, 7], the stress-strain equations

\[ Nx = A_{11} \varepsilon_x + A_{12} \varepsilon_y \]
\[ Ny = A_{12} \varepsilon_x + A_{22} \varepsilon_y \]
\[ N_{xy} = A_{66} \gamma_{xy} \]

averaged over the thickness H of the plate become

\[ \sigma_x = \frac{A_{11}}{H} \varepsilon_x + \frac{A_{12}}{H} \varepsilon_y \]
\[ \sigma_y = \frac{A_{12}}{H} \varepsilon_x + \frac{A_{22}}{H} \varepsilon_y \]
\[ \tau_{xy} = \frac{A_{66}}{H} \gamma_{xy} \]

where \( A_{ij} \) are plate membrane stiffnesses and the plate is assumed to have no stretching-bending coupling (symmetric layup) and no stretching-shearing coupling (\( A_{16} = A_{26} = 0 \)).
From the equilibrium equations

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0
\]

\[
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0
\]

one obtains by successive differentiation:

\[
\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_x}{\partial y^2}
\]  

(6)

\[
\frac{\partial^2 \sigma_y}{\partial y^2} = -\frac{\partial^2 \tau_{xy}}{\partial x \partial y}
\]  

(7)

Eqs (6) and (7) can be combined to yield,

\[
\frac{\partial^2 \sigma_y}{\partial y^2} = -\frac{\partial^2 \sigma_x}{\partial x^2}
\]  

(8)

Equations (6) and (8) can be placed into Eq. (4) to give:

\[
\left( \frac{A_{11} A_{22} - A_{12}^2}{A_{66}} - 2 A_{12} \right) \frac{\partial^2 \sigma_x}{\partial x^2} + A_{22} \frac{\partial^2 \sigma_x}{\partial y^2} + A_{11} \frac{\partial^2 \sigma_y}{\partial x^2 \partial y^2} = 0
\]

(9)

Differentiating Eq. (9) twice with respect to y,

\[
\left( \frac{A_{11} A_{22} - A_{12}^2}{A_{66}} - 2 A_{12} \right) \frac{\partial^4 \sigma_x}{\partial y^4} + A_{22} \frac{\partial^4 \sigma_x}{\partial y^2 \partial x^2} + A_{11} \frac{\partial^4 \sigma_y}{\partial x^2 \partial y^2} = 0
\]

(10)

Also, differentiating Eq. (8) twice with respect to x,

\[
\frac{\partial^4 \sigma_x}{\partial x^4} = -\frac{\partial^2 \sigma_x}{\partial x^2}
\]  

(11)

and substituting into Eq. (10), the final equation for \( \sigma_x \) is obtained:

\[
\frac{\partial^4 \sigma_x}{\partial x^4} + \left[ \frac{A_{11} A_{22} - A_{12}^2}{A_{66}} - 2 A_{12} \right] \frac{\partial^4 \sigma_x}{\partial y^4} + A_{22} \frac{\partial^4 \sigma_x}{\partial y^2 \partial x^2} + A_{11} \frac{\partial^4 \sigma_y}{\partial x^2 \partial y^2} = 0
\]

(12)

This equation is to be solved subject to the boundary conditions:

\[
\sigma_x(x = 0) = 0 \quad f o r \quad 0 \leq y \leq \frac{b - h}{2}
\]

(13)

\[
\frac{b + h}{2} \leq y \leq b
\]

\[
\sigma_x(x = a) = \sigma_o = \frac{F}{bH} \quad f o r \quad \frac{b - h}{2} \leq y \leq \frac{b + h}{2}
\]

(14)

\[
\sigma_y(y = 0) = \tau_{xy}(y = b) = 0
\]

(15)

\[
\tau_{xy}(x = 0) = \tau_{xy}(x = a) = \tau_{xy}(y = 0)
\]

(16)

\[
\tau_{xy}(y = b) = 0
\]

F is the total force acting on each edge of the plate and h is the width over which F is distributed on edge x = 0 of the plate. It is straightforward to verify that Eq. (12) accepts solutions of the form

\[
\sigma_x \approx f(x) \cos \frac{n \pi y}{b}
\]

(17)
Substituting in Eq. (12), \( f_n(x) \) is found to satisfy the following equation,

\[
\frac{d^4 f_n}{dx^4} - \beta \left( \frac{n\pi}{b} \right)^2 \frac{d^2 f_n}{dx^2} + \gamma \left( \frac{n\pi}{b} \right)^4 f_n = 0
\]

where

\[
\beta = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}A_{66}} - 2\frac{A_{12}}{A_{11}}
\]

\[
\gamma = \frac{A_{22}}{A_{11}}
\]

The solution to Eq. (18) is in the form \( f_n = D e^{\phi x} \) with

\[
\phi = \pm \frac{1}{\sqrt{2}} \left( \frac{n\pi}{b} \right) \sqrt{\beta \pm \sqrt{\beta^2 - 4\gamma}}
\]

If the plate were infinitely long the two solutions in Eq. (20) with positive real parts would be neglected to avoid infinite stresses. It will be assumed here that the plate is long enough so that the solution for an infinite plate is still valid. As will be shown later, for most practical cases this is sufficient. For very short plates, positive real parts would be neglected to avoid infinite stresses.

Using the solution for \( f_n \) to substitute in Eq. (17) and noting that Eq. (12) also accepts constants as solutions, the longitudinal stress \( \sigma_x \) in the plate can be written in its most general form as:

\[
\sigma_x = Ko + \sum_{n=1}^{\infty} A_n \left( e^{\phi_1 x} + C_n e^{\phi_2 x} \right) \cos \frac{n\pi y}{b}
\]

where \( Ko, A_n, \) and \( C_n \) are unknowns.

Using Eq. (21), the equilibrium Eqs. (4), and the boundary conditions (15) and (16), it is found that the index \( n \) must be even and the stresses are given by (substituting \( 2n \) for \( n \),

\[
\sigma_x = Ko + \sum_{n=1}^{\infty} A_n \left( e^{\phi_1 x} - \frac{\phi_1}{\phi_2} e^{\phi_2 x} \right) \cos \frac{2n\pi y}{b}
\]

\[
\sigma_y = \sum_{n=1}^{\infty} \frac{b}{2n\pi} \left( \frac{\phi_1}{\phi_2} - 1 \right) \phi_1 A_n \left( \phi_1 e^{\phi_1 x} - \phi_2 e^{\phi_2 x} \right) \sin \frac{2n\pi y}{b}
\]

The two remaining unknowns \( Ko \) and \( A_n \) are determined by satisfying the remaining boundary conditions (13) and (14). From the first of Eqs. (22), \( \sigma_x \) can be treated as a Fourier cosine series. Then, \( Ko \) must equal the average of \( \sigma_x \) at any \( x \) location,

\[
Ko = \frac{F}{bh}
\]

In practice, the series in Eqs. (22) are truncated. The number of terms to be used is selected by comparing the distribution of \( \sigma_x \) at \( x = 0 \) with the exact distribution that reproduces the concentrated load at \( x = 0 \). This is done in Figure 2 for the case of a (±45°) layup with properties given in Table 1. Stress is normalized with the uniform stress at \( x = a \). For the purposes

FIG. 2. Stress at the panel edge where concentrated load is applied as a function of number of terms in the series.
of comparison, the dimensions of the panel are $a = 50.8$ cm (20 in), $b = 15.24$ cm (6 in) and the concentrated load is over a distance $h = 1.778$ cm (0.7 in). The applied force $F$ is 8900 N (2000 lb).

As can be seen from Figure 2, the exact shape of the stress distribution requires a large number of terms. Even for $n = 160$ there is a small difference between the exact distribution and the prediction. On the other hand, the difference between $n = 80$ and $n = 160$ is small enough that it was decided to use $n = 80$ in all cases from now on. As will be shown later with comparison to a finite element solution, this provides sufficient accuracy in all cases examined.

### 3. FINITE ELEMENT ANALYSIS PREDICTION

To validate the closed form solution for stress distribution within the composite plate, a fine mesh finite element model was created. Panel dimensions of 15.2 cm (6 inches) in width and 50.8 cm (20 inches) in length were used. (The 50.8 cm length was aligned with the X axis of the global cartesian coordinate system). ANSYS v.11 was used to generate a fine mesh with the focus of using the linear static analysis solution. First order isoparametric shell elements were used. Element edge lengths of .025 to .175 in. were used to transition a refined mesh near the load application to coarse density near the far-field constraint boundary condition. This generated 37,282 SHELL181 elements and 223,692 degrees of freedom. The model and applied loading were configured as shown in Figure 3.

The finite element model was generated with a very fine mesh in the load application area in order to accurately capture the stresses in the high gradient region. Such a fine mesh was necessary for more accurate comparison to the analytical solution. Subsequently, matching the analytical method presented here to the finite element results would increase the confidence in the analytical model.

The element properties were the same layup, ($\pm 45$)/$(0/90)/$ ($\pm 45$), with ply material properties as defined in Table 1. Element material coordinate directions were defined to be aligned with the global cartesian X-Y system. ANSYS shell section definition was used to define the ply layup stacking sequence with respect to the element material coordinate system. An applied load of 4445 N (1,000 lb) was used acting in-plane over a discrete 1.78 cm (0.70 in) edge length representing the load introduction from a typical stringer flange. The displacement boundary conditions were set such that the panel reacts to the

![FIG. 3. Fine mesh finite element model, applied loads, and boundary conditions.](image-url)
applied load uniformly in the X direction and is able to expand naturally due to Poisson’s effect in the Y direction.

The ANSYS linear static solution was performed. The results and boundary conditions were checked to verify the model reproduced a $\sigma_1$ stress on the applied load edge and confirmed proper level of uniform far-field stress sufficiently far from the boundary conditions. Checks also were performed to confirm far-field total load reaction. The figures presented here (Figures 4–6) are contour plots demonstrating stress distribution within the composite panel. It is interesting to observe, in the figures, the panel deflection as well as the localized dissipation of stresses directly beyond the load introduction. Detailed numerical comparison of the finite element results with the proposed analytical solution is presented in the following section.

Additional finite element model verification was performed to assess the adequacy of the mesh density. Two additional coarse mesh models were run to converge on an element size that would produce the most confidence in the results. Table 2 compares the smallest element mesh size near the applied load area along with peak normal, transverse, and shear stress results. In each case, the stress values are normalized by the peak values of the previous mesh. The final column in Table 2 shows that the final mesh density has essentially converged on the best practical mesh for this study. It is interesting to note that the peak shear stress converges more slowly than the other two stresses in Table 2.

4. CORRELATION WITH FINITE ELEMENT ANALYSIS

The normalized axial stress along the center of the panel is compared to the finite element model prediction for the ($\pm 45$)/(0/90)/($\pm 45$) layup in Figure 7. The finite
element results are in excellent agreement with the analytical solution.

The comparison of the two prediction methods for transverse stress as a function of normalized transverse distance at the \( x = 0.381 \) cm (0.15 in) location is shown in Figure 8. This particular location was chosen based on the element size used in the finite element model. A path plot through the finite element model at this section was used to extract stress results for correlation. This provides several rows of elements away from the load introduction point to minimize the effects of point-load boundary conditions. Using element center results as opposed to grid point averaged results provides more accuracy as the results are calculated using shape functions from the element’s exact integration points. The analytical and finite element methods are in good agreement. The analytical solution predicts a peak that is 12% higher than that of the finite element solution. Also, there are two short plateaus in the finite element solution on either side of the peak, that the analytical solution does not accurately predict. It is believed that using more terms in the series of the analytical solution would improve the accuracy in this case.

<table>
<thead>
<tr>
<th>TABLE 2</th>
<th>Mesh density study</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Starting mesh A</td>
</tr>
<tr>
<td>Smallest element edge length (mm)</td>
<td>2.540</td>
</tr>
<tr>
<td>Number of elements</td>
<td>7109</td>
</tr>
<tr>
<td>Peak axial stress</td>
<td>1.000</td>
</tr>
<tr>
<td>Peak transverse stress</td>
<td>1.000</td>
</tr>
<tr>
<td>Peak shear stress</td>
<td>1.000</td>
</tr>
</tbody>
</table>

FIG. 6. Finite element model shear stress contour plot.

FIG. 7. Normalized axial stress \( \sigma_x bH/F \) along the center of the panel.

FIG. 8. Normalized transverse stress \( \sigma_y bH/F \) as a function of distance \( y \) evaluated at \( x/a = 0.0075 \).
The shear stress comparison is shown in Figure 9 as a function of distance across the panel at the same x location, $x = 0.381$ cm (0.15 in). The analytical method proposed here is in excellent agreement with the finite element results.

5. RESULTS

The excellent agreement with the finite element results gives confidence in the present method, which can be used to obtain the stress distributions for different plate layups and geometries. This will give an idea of the size and layup of reinforcement necessary to transfer the load from a stiffened panel to a monolithic or sandwich panel. Typical results are presented below for three different panel layups, $(\pm 45)_{4}$, (0/90)$_{4}$, and the quasi-isotropic $[(\pm 45)/(0/90)]_{s}$. These layups are fairly typical of sandwich panels where each facesheet of the sandwich consists of two of the four plies specified. The core type and thickness do not affect the in-plane stress distribution. The panel length “a” is 50.8 cm (20 in), the width “b” is 15.24 cm (6 in) and the concentrated load of 2000 lb is over a distance $h = 1.778$ cm (0.7 in).

The axial stress $\sigma_{x}$ along x at the center of the panel is shown in Figure 10. The peak stress is the same for all three layups since the layup thickness, applied load and the width over which it acts, are all the same. As x increases however, the rate at which the stress decreases depends strongly on the layup. The $(\pm 45)_{4}$ has the steepest rate of decay, followed by the quasi-isotropic layup. The (0/90)$_{4}$ layup, which has no 45 degree plies, has the slowest rate of decay. At the far field, ($x$ large) all three curves go to the uniform stress at the $x = a$ edge of the panel. However, because of the different decay rates, the reinforcement for the (0/90)$_{4}$ layup, needed to accommodate the stresses in excess of the far-field stresses, should be significantly longer that the other two. Assuming that a reinforcement that covers the region over which the maximum stress drops to within 20% of the far field value is adequate, the length of the reinforcement would correspond to $x/a = 0.45$ for this layup while it would correspond to $x/a < 0.3$ for the other two cases.

The transverse stress $\sigma_{y}$ is shown in Figure 11 as a function of y at $x = 2.54$ mm (0.1 in) from the $x = 0$ edge. Here, the $(\pm 45)_{4}$ layup shows a peak that is significantly higher than the value of the other two layups. So while the axial stress $\sigma_{x}$ for the $(\pm 45)_{4}$ layup was shown to die down to its far-field value much faster than for the (0/90)$_{4}$ layup and thus require less of a transition region, the $\sigma_{y}$ stress is twice as high as for the (0/90)$_{4}$ layup suggesting that the $(\pm 45)_{4}$ layup may be more critical. The peak value of this stress should also be taken into account when the reinforcement is designed.

Finally, the shear stress $\tau_{xy}$ is shown in Figure 12 as a function of y at the same x location ($x/a = 0.005$). As in the case of transverse stress, the $(\pm 45)_{4}$ layup has higher stresses than the other two layups. Again, the peak stresses can be significant and should be accounted for in the design of the reinforcement.
Similar plots can be obtained for the axial stress $\sigma_x$ versus $y$ (not shown here for conciseness). The corresponding results at other $x$ locations (also not shown), show similar distributions but the peaks decrease significantly as $x$ increases. Therefore, a weight-optimum local reinforcement should have thickness and layup tapering both in the $x$ and $y$ directions and should be such that the combination of $\sigma_x$, $\sigma_y$, and $\tau_{xy}$ does not lead to failure at any given location. The dimensions of the reinforcement can be obtained from graphs like those in Figures 10 through 12 as the distances over which the stresses are significantly higher than the far-field stresses. For example, on the basis of the results in Figures 11 and 12, the transverse and shear stresses are confined in a region $0.35 \leq y/b \leq 0.65$. This would suggest a doubler width $dy$ such that $dy/y = (0.65-0.35)/0.3$, which, for the present case, leads to a value $dy = 4.57$ cm (1.8 in). This value also covers the $\sigma_x$ distribution for these three layups. It is important to note that this value of $dy$ is significantly larger than the width $h$ (= 1.778 cm or 0.7 in) over which the applied load is distributed. Thus, in the case of a terminated stiffener, creating a local reinforcement with width equal to the stiffener flange width may not be sufficient. In addition, the rate of decay of $\sigma_x$ in Figure 10 suggests a reinforcement length $dx$ such that $dx = (0.5 a)$ for $(0/90)_4$ and $0.3a$ for $(\pm 45)_4$ or $[(\pm 45)/(0/90)]_s$ layups (giving $dx = 25.4$ cm or 10 in and $dx = 15.24$ or 6 in respectively). These should be viewed as guiding examples only. The exact dimensions and layup of the reinforcement should be decided upon, detailed analysis of the plate with the local reinforcement in place is necessary.

One final comment about the effect of plate length or stiffener spacing is in order. As long as the plate is long enough so the stresses in Figure 10 have decayed close to their far-field values before reaching the other end, the solution presented is valid. If the plate is shorter than that, (shorter than $0.5a$ for $(0/90)_4$) the solution must be modified and four exponentials using all values of $\phi$ in Eq. (20) should be used. Also, if the stiffener spacing is greater than the range $dy$ determined based on Figures 8 and 9 (0.3 b), the stress distribution caused by one stiffener is not affected by the adjacent stiffeners and the solution is valid. If this is not the case, the solution should be modified to account for multiple load introduction points. The approach is the same as before and only the expression of the coefficients $A_n$ changes as the boundary condition (13) at $x = 0$ must reflect the presence of more than one stiffener introducing load into the panel.

6. SUMMARY AND CONCLUSIONS

An approach to determine the in-plane stresses in an anisotropic plate loaded by a concentrated load at one end and uniform load distributed over the whole edge at the opposite end was presented. The method gives the stresses in closed form and was found to be in excellent agreement with very detailed finite element results. One of the advantages of the method is that it can be used to obtain preliminary design (dimensions and layup) of the reinforcements necessary to transfer the concentrated load into the plate without failure. This is preliminary as the presence of the reinforcement will alter the load locally and the solution must be iterated upon to obtain the final design for the reinforcement.

It was found that the axial stresses in the panel decay more slowly for $(0/90)$ panels than for $(\pm 45)$ or quasi-isotropic panels. On the other hand, the peak transverse and shear stresses in the $(\pm 45)$ panels are significantly higher than in the quasi-isotropic or $(0/90)$ panels. These peaks must be accounted for in the reinforcement design to avoid failure.

REFERENCES